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Generalised Measures of Non-precompactness Relative to von Neumann Algebras

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Suppose $\mathcal{M}$ is a von Neumann algebra on a Hilbert space $\mathcal{H}$ and $\mathcal{J}$ is any norm closed ideal in $\mathcal{M}$. We will characterise $\mathcal{J}$ in terms of a generalised measure of non-precompactness on $\mathcal{H}$ which, in the case where $\mathcal{M}$ is the algebra of all bounded operators on $\mathcal{H}$ and $\mathcal{J}$ is the ideal of compact operators, reduces to the classical ball measure of non-compactness. Now suppose $\mathcal{M}$ is a semifinite von Neumann algebra, τ a faithful semifinite normal trace on $\mathcal{M}$, and $\tilde{\mathcal{M}}$ the space of τ-measurable operators. We find generalised measures of non-precompactness on $\mathcal{H}$ which are parametrised by the positive half-line and which can be used to characterise the ideal of τ-compact operators. © 1995 Academic Press, Inc.

In analysis the notion of "smallness" is a crucial one. The classical class of small sets is the collection of relatively compact sets in a Banach space. Here relative compactness and total boundedness coincide, that is, a set A is relatively compact if and only if for all $\varepsilon > 0$ there exists a finite set F such that $A \subset F + \varepsilon B_{\mathcal{E}}$. (Here $B_{\mathcal{E}}$ denotes the unit ball of $\mathcal{E}$.) As further examples, similar characterisations hold for the collection of weakly compact sets in a Banach space and the almost order bounded sets in a Banach lattice. A broad approach to the idea of smallness is that taken in [2, 1] with the notion of $\mathcal{B}$-precompactness in the setting of a topological vector space: given a vector bornology $\mathcal{B}$, a set A is $\mathcal{B}$-precompact if for every neighbourhood U of 0 there is some $B \in \mathcal{B}$ such that $A \subset B + U$.

In the literature, there are examples of "measures of non-smallness" given a suitable class of small sets: the canonical measure of non-compactness, de Blasi's measure of non-weak-compactness in a Banach space [3], and the measure of de Pagter and Schep in the setting of Banach lattices

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[10] are examples of this. In the general case mentioned, where the topology on the vector space is assumed to be metrisable, one can define a measure of non-$\mathcal{B}$-precompactness which would measure the deviation of a set from the $\mathcal{B}$-precompact sets. In the first section we make some simple preparatory remarks about this measure in the normed space setting

In the second section we consider the case of a von Neumann algebra $\mathcal{M}$ acting on a Hilbert space $\mathcal{H}$ and containing a norm closed ideal $\mathcal{J}$. We then consider the bornology $\mathcal{B} = b(\mathcal{J}^p)$ which was introduced in [17]. The measure of non-$\mathcal{B}$-precompactness in this setting is then the function $q_\mathcal{J}$ considered by Ströh and Swart [11, 12]. In the case that $\mathcal{M}$ is the von Neumann algebra of all bounded operators on $\mathcal{H}$ and $\mathcal{J}$ is the ideal of compact operators, $q_\mathcal{J}$ reduces to the ball measure of non-compactness. We find a minimax formulation of the function $q_\mathcal{J}$ which is used to derive properties of $q_\mathcal{J}$ analogous to those of the ball measure of non-compactness.

In the third section of this paper we consider the case where $\mathcal{M}$ is semifinite and τ is a faithful semifinite normal trace on $\mathcal{M}$. We denote by $\tilde{\mathcal{M}}$ the space of all τ-measurable operators, and consider the measure closed ideal $\tilde{\mathcal{M}}$ of τ-compact operators. For $t > 0$ we find functions q_t on $\mathcal{H}$ which can be viewed as being parametrised measures of non-compactness (or as a parametrised version of $q_\mathcal{J}$). In fact the functions again admit minimax formulations, and in the case that $\mathcal{M}$ is the von Neumann algebra of all bounded operators on $\mathcal{H}$ and τ is the canonical trace, the function q_t is constant on the intervals $[n, n + 1)$ for $n = 0, 1, 2, \ldots$ and q_n coincides with the nth-width function first defined by Kolmogorov [7; 6, II, Sect. 2.4]. Our measures are explicitly related to the generalised singular functions of the τ-measurable operators. A key role is played here by the measure topology on the underlying Hilbert space, as introduced by Nelson [9].

1. The Measure of Non-$\mathcal{B}$-precompactness

Suppose $\mathcal{E}$ is a normed space. We shall call a collection of subsets of $\mathcal{E}$ which is closed under finite sums, scalar multiples, and balanced hulls, and hereditary under inclusion, a vector bornology. If the collection is in addition closed under convex hulls then it is called a convex bornology. A subset $\mathcal{B}_0$ of a bornology $\mathcal{B}$ is a base for $\mathcal{B}$ if every $B \in \mathcal{B}$ is contained in some $B_0 \in \mathcal{B}_0$. For every vector bornology $\mathcal{B}$ we can form a new bornology $\tilde{\mathcal{B}}$ consisting of all subsets A of $\mathcal{E}$ such that for every $\varepsilon > 0$ there is a $B \in \mathcal{B}$ such that $A \subset B + \varepsilon B_\mathcal{E}$. (These sets are called $\mathcal{B}$-precompact in [1].) It is easy to see that if $\mathcal{B}$ is convex then so is $\tilde{\mathcal{B}}$, and if an operator $x : \mathcal{E} \to \mathcal{E}$ is norm bounded and $\mathcal{B}$ to $\mathcal{B}$ bounded then it is $\tilde{\mathcal{B}}$ to $\tilde{\mathcal{B}}$ bounded. We can consider the measure of non-$\mathcal{B}$-precompactness

$$q_{\mathcal{B}}(B) = \inf\{\varepsilon > 0 : \exists F \in \mathcal{B} \text{ such that } B \subset F + \varepsilon B_{\mathscr{E}}\}. \tag{1}$$

It is clear that $q_{\mathcal{B}}(B) = 0$ iff $B \in \tilde{\mathcal{B}}$. In the above equation we may consider the sets F as coming from any base $\mathcal{B}_0$ for $\mathcal{B}$. Furthermore, the reader may easily verify the following properties of $q_{\mathcal{B}}$.

Suppose $B, B_1, B_2 \subset \mathscr{E}$ and $\lambda \in \mathbf{C}$.

(i) $q_{\mathcal{B}}(B_1) \leq q_{\mathcal{B}}(B_2)$ if $B_1 \subset B_2$;

(ii) $q_{\mathcal{B}}(\lambda B) = |\lambda| q_{\mathcal{B}}(B)$ (with the convention that $0 \cdot \infty = 0$);

(iii) $q_{\mathcal{B}}(\text{bal } B) = q_{\mathcal{B}}(B)$; $q_{\mathcal{B}}(\text{ac } B) = q_{\mathcal{B}}(B)$ if $\mathcal{B}$ is convex (here bal refers to the balanced hull and ac to the absolutely convex hull);

(iv) $q_{\mathcal{B}}(B_1 + B_2) \leq q_{\mathcal{B}}(B_1) + q_{\mathcal{B}}(B_2)$;

(v) $q_{\mathcal{B}}(B_{\mathscr{E}}) \leq 1$;

(vi) $q_{\mathcal{B}}(xB) \leq \|x\| q_{\mathcal{B}}(B)$ if $x : \mathscr{E} \to \mathscr{E}$ is norm-bounded and $\mathcal{B}$ to $\mathcal{B}$ bounded.

PROPOSITION 1.1. *Suppose $\mathcal{B}$ is a bornology on $\mathscr{E}$ and $\mathcal{A}$ is an algebra of operators on $\mathscr{E}$ which are norm bounded and $\mathcal{B}$ to $\mathcal{B}$ bounded. Then*

(i) *$j(\mathcal{B}) = \{x \in \mathcal{A} : xB_{\mathscr{E}} \in \mathcal{B}\}$ is an ideal of $\mathcal{A}$;*

(ii) *$i(\mathcal{B}) = \{x \in \mathcal{A} : xB_{\mathscr{E}} \in \tilde{\mathcal{B}}\} = \{x \in \mathcal{A} : q_{\mathcal{B}}(xB_{\mathscr{E}}) = 0\}$ is a norm closed ideal of $\mathcal{A}$.*

Proof. It is easy to verify that $j(\mathcal{B})$ is an ideal in $\mathcal{A}$. Since $i(\mathcal{B}) = j(\tilde{\mathcal{B}})$ we have that $i(\mathcal{B})$ is also an ideal. We show that it is closed. Suppose $i(\mathcal{B}) \supset x_n \xrightarrow{\|\cdot\|} x \in \mathcal{A}$. Given $\varepsilon > 0$, find n sufficiently large so that $\|x - x_n\| \leq \varepsilon$ and then find $F \in \mathcal{B}$ such that $x_n B_{\mathscr{E}} \subset F + \varepsilon B_{\mathscr{E}}$. Then $xB_{\mathscr{E}} \subset F + 2\varepsilon B_{\mathscr{E}}$, which shows that $x \in i(\mathcal{B})$. ∎

2. THE FUNCTION $q_{\mathcal{J}}$

In the sequel $\mathcal{M}$ will be a von Neumann algebra of bounded linear operators on a Hilbert space $\mathcal{H}$, and $\mathcal{M}^p$ the complete lattice of orthogonal projections in $\mathcal{M}$. $\mathcal{J}$ will denote a norm closed two-sided (equivalently, self-adjoint) ideal in $\mathcal{M}$. The set $\mathcal{J}^p = \mathcal{J} \cap \mathcal{M}^p$ forms what is called a p-ideal [19, 18], that is, it is closed under the formation of joins, subprojections, and equivalent projections. The closed unit ball of $\mathcal{H}$ will be denoted $B_{\mathcal{H}}$.

The function $q_{\mathcal{J}}$ was first considered by Ströh and Swart; see [11, Sect. 3; 12, Definition 3.1]. Here we view $q_{\mathcal{J}}$ as the generalised measure of non-precompactness determined by the convex bornology $b(\mathcal{J}^p)$. This bornology, discussed in [17], has as a base the sets $\{p\delta B_{\mathcal{H}} : p \in \mathcal{J}^p, \delta > 0\}$. It follows from [17, Corollary 2.9 and Note 3.3] or by direct calculation, that every $x \in \mathcal{M}$ is $b(\mathcal{J}^p)$ to $b(\mathcal{J}^p)$ bounded. We give an explicit definition.

DEFINITION 2.1. Suppose $B \subset \mathcal{H}$. The measure of non-$\mathcal{J}$-precompactness of B is

$$q_{\mathcal{J}}(B) = \inf\{\varepsilon > 0 : \exists p \in \mathcal{J}^p,\ \delta > 0 \text{ such that } B \subset p\delta B_{\mathcal{H}} + \varepsilon B_{\mathcal{H}}\}. \quad (2)$$

B is said to be $\mathcal{J}$-precompact if $q_{\mathcal{J}}(B) = 0$.

In the case that $\mathcal{M}$ is the algebra of all bounded linear operators on $\mathcal{H}$ and $\mathcal{J}$ is the ideal of compact operators, $q_{\mathcal{J}}$ reduces to the canonical ball measure of non-compactness. The following proposition summarises the properties of $q_{\mathcal{J}}$ that follow from Section 1.

PROPOSITION 2.2. *Suppose $B, B_1, B_2 \subset \mathcal{H}$, $\lambda \in \mathbf{C}$, $x \in \mathcal{M}$.*

$$B_1 \subset B_2 \Rightarrow q_{\mathcal{J}}(B_1) \le q_{\mathcal{J}}(B_2) \quad (3)$$

$$q_{\mathcal{J}}(\lambda B) = |\lambda| q_{\mathcal{J}}(B) \quad (4)$$

$$q_{\mathcal{J}}(B) = q_{\mathcal{J}}(\mathrm{ac}\, B) \quad (5)$$

$$q_{\mathcal{J}}(B_1 + B_2) \le q_{\mathcal{J}}(B_1) + q_{\mathcal{J}}(B_2) \quad (6)$$

$$q_{\mathcal{J}}(xB) \le \|x\| q_{\mathcal{J}}(B) \quad (7)$$

$$q_{\mathcal{J}}(xB) = q_{\mathcal{J}}(|x|B). \quad (8)$$

PROPOSITION 2.3. *Suppose $B \subset \mathcal{H}$. Then $q_{\mathcal{J}}(B) < \infty$ iff B is norm bounded. If B is norm bounded then $q_{\mathcal{J}}(B) \le \varepsilon$ iff there exists $p \in \mathcal{J}^p$ such that $B \subset p(B) + \varepsilon B_{\mathcal{H}}$.*

Proof. It is clear that $q_{\mathcal{J}}(B) < \infty$ iff B is norm bounded. So suppose that B is norm bounded, and suppose that $q_{\mathcal{J}}(B) \le \varepsilon$, so there exists $p \in \mathcal{J}^p$ and $\delta > 0$ such that $B \subset p\delta B_{\mathcal{H}} + \varepsilon B_{\mathcal{H}}$. Then indeed $\|\xi - p\xi\| = \inf_{\eta \in p\mathcal{H}} \|\xi - \eta\| \le \inf_{\eta \in p\delta B_{\mathcal{H}}} \|\xi - \eta\| \le \varepsilon$ for all $\xi \in B$. Thus $B \subset p(B) + \varepsilon B_{\mathcal{H}}$.
The converse is clear: $p(B)$ is the required member of $b(\mathcal{J}^p)$—this set is bounded in norm since B is bounded in norm. ∎

The following minimax characterisation of the function $q_{\mathcal{J}}$ follows from the previous proposition. For any $B \subset \mathcal{H}$

$$q_{\mathcal{J}}(B) = \begin{cases} \inf\limits_{p \in \mathcal{J}^p} \sup\limits_{\xi \in B} \|\xi - p\xi\| = \inf\limits_{p \in 1 - \mathcal{J}^p} \sup\limits_{\xi \in B} \|p\xi\| & \text{if } B \text{ is norm bounded} \\ \infty & \text{otherwise.} \end{cases} \quad (9)$$

We now establish some further elementary properties of $q_{\mathcal{I}}$ which are also analogous to the properties enjoyed by the canonical ball measure of non-compactness: see, for example, [8, Proposition 4.13; 4, I, Lemma 2.2].

PROPOSITION 2.4. *Suppose* $B, B_1, B_2 \subset \mathcal{H}$, $\lambda \in \mathbf{C}$.

$$q_{\mathcal{I}}(B_{\mathcal{H}}) = 1 \tag{10}$$

$$q_{\mathcal{I}}(B_1 \cup B_2) = \max\{q_{\mathcal{I}}(B_1); q_{\mathcal{I}}(B_2)\} \tag{11}$$

$$q_{\mathcal{I}}(\overline{B}) = q_{\mathcal{I}}(B). \tag{12}$$

Proof. Equation (10) follows easily from (9). (Here we are implicitly assuming that $\mathcal{I} \neq \mathcal{M}$. In the case that $\mathcal{I} = \mathcal{M}$, $q_{\mathcal{I}}$ vanishes on bounded sets.)

We may suppose that the quantities involved in (11) and (12) are finite, otherwise it is easy to see that there is nothing to prove.

It follows from (3) that $q_{\mathcal{I}}(B_1 \cup B_2) \geq \max\{q_{\mathcal{I}}(B_1); q_{\mathcal{I}}(B_2)\}$. Conversely, suppose that for $i = 1, 2$ we have $p_i \in \mathcal{I}^P$ and $\delta_i > 0$ such that $B_i \subset p_i\delta_i B_{\mathcal{H}} + \varepsilon_i B_{\mathcal{H}}$. Then

$$\begin{aligned} B_1 \cup B_2 &\subset (p_1\delta_1 B_{\mathcal{H}} + \varepsilon_1 B_{\mathcal{H}}) \cup (p_2\delta_2 B_{\mathcal{H}} + \varepsilon_2 B_{\mathcal{H}}) \\ &\subset (p_1\delta_1 B_{\mathcal{H}} \cup p_2\delta_2 B_{\mathcal{H}}) + \max\{\varepsilon_1, \varepsilon_2\} B_{\mathcal{H}} \\ &\subset (p_1 \vee p_2)\max\{\delta_1, \delta_2\} B_{\mathcal{H}} + \max\{\varepsilon_1, \varepsilon_2\} B_{\mathcal{H}}. \end{aligned}$$

Hence $q_{\mathcal{I}}(B_1 \cup B_2) \leq \max\{\varepsilon_1, \varepsilon_2\}$, and (11) follows.

Since $B \subset \overline{B}$, we have from (3) that $q_{\mathcal{I}}(B) \leq q_{\mathcal{I}}(\overline{B})$. Conversely, if $B \subset p\delta B_{\mathcal{H}} + \varepsilon B_{\mathcal{H}}$, then $\overline{B} \subset p\delta B_{\mathcal{H}} + \varepsilon B_{\mathcal{H}} + \eta B_{\mathcal{H}} = p\delta B_{\mathcal{H}} + (\varepsilon + \eta) B_{\mathcal{H}}$ for any $\eta > 0$. Thus (12) follows. ∎

Let $\pi : \mathcal{M} \to \mathcal{M}/\mathcal{I} : x \to x + \mathcal{I}$ be the canonical quotient map. It is well known, see, for example, [14, I.8.1], that $\mathcal{M}/\mathcal{I}$ is a $\mathcal{C}^*$-algebra when equipped with the canonical quotient norm and involution $\pi(x)^* = \pi(x^*)$. We denote by α the seminorm on $\mathcal{M}$ induced by the quotient norm on $\mathcal{M}/\mathcal{I}$, i.e., $\alpha(x) = \|\pi(x)\| = \inf_{y \in \mathcal{I}} \|x - y\|$. It is well known that α admits the characterisations

$$\begin{aligned} \alpha(x) &= \inf\{\lambda \geq 0 : e_{(\lambda,\infty)}(|x|) \in \mathcal{I}^P\} & (13) \\ &= \inf\{\|px\| : p \in 1 - \mathcal{I}^P\}. & (14) \end{aligned}$$

Here $e_B(\cdot)$ denotes the spectral projection corresponding to the Borel set $B \subset \mathbf{R}$, and instead of $e_{(-\infty,t]}(\cdot)$ we will just write $e_t(\cdot)$.

Equation (15) was first shown using different techniques by Ströh and Swart [12, Corollary 3.8; 11, Theorem 3.9]. It is viewed as a generalisation of the canonical measure of non-compactness of an operator x: see [4, I, Sect. 2; 8, Sect. 3].

THEOREM 2.5. *Suppose $B \subset \mathcal{H}$ is bounded in norm and $x \in \mathcal{M}$, $y \in \mathcal{J}$.*

$$q_{\mathcal{J}}(xB_{\mathcal{H}}) = \alpha(x) \tag{15}$$

$$q_{\mathcal{J}}(xB) \leq \alpha(x)q_{\mathcal{J}}(B) \tag{16}$$

$$q_{\mathcal{J}}(xB) = q_{\mathcal{J}}((x + y)B). \tag{17}$$

Proof. For (15), $q_{\mathcal{J}}(xB_{\mathcal{H}}) = \inf_{p \in 1-\mathcal{J}^p} \sup_{\xi \in B_{\mathcal{H}}} \|px\xi\| = \inf_{p\in 1-\mathcal{J}^p} \|px\| = \alpha(x)$, by (14).

Suppose $\delta > 0$. Let $e = e_{\alpha(x^*)+\delta}(|x^*|)$. Then $\|ex\| = \|x^*e\| \leq \alpha(x^*) + \delta = \alpha(x) + \delta$ and by (13) we have that $1 - e \in \mathcal{J}^p$.

Suppose $f \in 1 - \mathcal{J}^p$. Let $g = N((1 - f)x^*) = 1 - R(x(1 - f))$ and let $p = e \wedge g$. Then $px = pgx = pgxf = pexf$. Now $1 - g = R(x(1 - f)) \sim R((1 - f)x^*) \leq 1 - f$ and so $1 - g \in \mathcal{J}^p$. Thus $1 - p = 1 - e \wedge g = (1 - e) \vee (1 - g) \in \mathcal{J}^p$, and so

$$q_{\mathcal{J}}(xB) \leq \sup_{\xi\in B} \|px\xi\| = \sup_{\xi\in B} \|pexf\xi\| \leq \sup_{\xi\in B} \|p\| \, \|ex\| \, \|f\xi\| \leq [\alpha(x) + \delta] \sup_{\xi\in B} \|f\xi\|.$$

Since $f \in 1 - \mathcal{J}^p$ was arbitrary and $\delta > 0$ was arbitrary, (16) follows. This argument is inspired by one of Nelson's: see [9, Theorem 1, Proof of (17′)]; or [15, Proof of Proposition 5(ii)]. Finally

$$\begin{aligned} q_{\mathcal{J}}((x + y)B) &\leq q_{\mathcal{J}}(xB + yB) \leq q_{\mathcal{J}}(xB) + q_{\mathcal{J}}(yB) \\ &\leq q_{\mathcal{J}}(xB) + \alpha(y)q_{\mathcal{J}}B) = q_{\mathcal{J}}(xB) \end{aligned}$$

and the reverse inequality follows likewise, proving (17). ∎

Under the convention that $0 \cdot \infty = 0$, the requirement that B be bounded is necessary in (16): it is possible to find an unbounded set B and an operator x belonging to the ideal in question such that $q_{\mathcal{J}}(x(B)) = \infty$; see Example 3.8. Therefore in proving (17), having used (3) and (6), and also (16), we need the assumption that B be bounded.

COROLLARY 2.6. $x \in \mathcal{J} \Leftrightarrow xB_{\mathcal{H}} \in b(\tilde{\mathcal{J}}^p) \Leftrightarrow q_{\mathcal{J}}(xB_{\mathcal{H}}) = 0 \Leftrightarrow q_{\mathcal{J}}(xB) = 0$ *for norm bounded $B \subset \mathcal{H}$.*

3. AN APPLICATION TO τ-MEASURABLE OPERATORS

In this section we suppose $\mathcal{M}$ is a semifinite von Neumann algebra and τ a faithful semifinite normal trace on $\mathcal{M}$. A closed densely defined operator x is said to be affiliated to $\mathcal{M}$ if $u^*xu = x$ for every unitary operator u

belonging to the commutant $\mathcal{M}'$ of $\mathcal{M}$. An affiliated operator x is called τ-measurable if for every $t > 0$ there exists a $p \in \mathcal{M}^p$ such that $p\mathcal{H} \subset D(x)$ and $\tau(1 - p) \leq t$. The set of all τ-measurable operators, denoted by $\tilde{\mathcal{M}}$, is a *-algebra of operators on $\mathcal{H}$ where the sum and product operation is the closure of the ordinary sum and product.

The sets $\tilde{\mathcal{M}}(\varepsilon, t) = \{x \in \tilde{\mathcal{M}} : \exists p \in \mathcal{M}^p$ such that $\|xp\| \leq \varepsilon$ and $\tau(1 - p) \leq t\}$ form a base at 0 (as $\varepsilon, t > 0$ vary) for a metrisable vector topology τ_{cm} on $\tilde{\mathcal{M}}$, called the topology of convergence in measure. Equipped with this topology, $\tilde{\mathcal{M}}$ is a complete topological *-algebra in which $\mathcal{M}$ is dense. For $x \in \tilde{\mathcal{M}}$ the generalised singular function

$$\mu_t(x) = \inf\{\|xp\| : p \in \mathcal{M}^p, \tau(1 - p) \leq t\} \tag{18}$$

is finite valued and decreasing. Moreover, $x_n \xrightarrow{\tau_{cm}} x \Leftrightarrow \mu_t(x_n - x) \to 0$ for all $t > 0$. If $p \in \mathcal{M}^p$ then $\mu_t(p) = \chi_{(0,\tau(p))}(t)$. If $x \in \tilde{\mathcal{M}}$ then $e_t(|x|) \xrightarrow{\tau_{cm}} 1$ as $t \to \infty$. For proofs of these facts the reader may consult [5, 15].

Since the generalised singular function is decreasing, we can define $\mu_\infty(x) = \lim_{t\to\infty} \mu_t(x)$. The ideal of τ-compact operators, denoted $\tilde{\mathcal{M}}_0$, consists of those $x \in \tilde{\mathcal{M}}$ for which $\mu_\infty(x) = 0$. Furthermore, the set of projections having finite trace forms a p-ideal which is denoted $\mathcal{P}_\tau$. The norm closed ideal generated by $\mathcal{P}_\tau$ is $\mathcal{M}_0 = \tilde{\mathcal{M}}_0 \cap \mathcal{M}$.

For more details on the τ-compact operators, see [5, 13, 16]. Here we find parametrised measures of non-precompactness relative to $\tilde{\mathcal{M}}_0$. As we shall see, it is appropriate to consider $\mathcal{H}$ to be equipped with the topology of convergence in measure as defined by Nelson [9]. Recall that this vector topology (which we also denote by τ_{cm}, as there is no danger of confusion) has as basic neighbourhoods of 0 the sets $\mathcal{H}(\varepsilon, t) = \{\xi \in \mathcal{H} : \exists p \in \mathcal{M}^p$ such that $\|p\xi\| \leq \varepsilon$, $\tau(1 - p) \leq t\}$, where $\varepsilon, t > 0$ are allowed to vary. We will also have occasion to consider such sets where $\varepsilon = 0$ or $t = 0$, and it is easy to see that $\mathcal{H}(0, t) = \bigcup \{p\mathcal{H} : p \in \mathcal{M}^p, \tau(p) \leq t\}$ and $\mathcal{H}(\varepsilon, 0) = \varepsilon B_{\mathcal{H}}$, although we stress that these sets are not neighbourhoods of the topology.

Recall that Nelson represented $\tilde{\mathcal{M}}$ as an algebra of (everywhere defined) operators on the abstract completion of $(\mathcal{H}, \tau_{cm})$. Of course $\tilde{\mathcal{M}}$ has a concrete realisation (as shown by Marianne Terp in [15], and in fact by Nelson—see the remarks at the end of Section 2 in [9])—but this is not the case for the completion of $\mathcal{H}$. In fact, Terp defines $\tilde{\mathcal{M}}$ without any reference to τ_{cm} on $\mathcal{H}$. For our purposes a middle road is most suitable: we make full use of τ_{cm} on $\mathcal{H}$ but do not consider the completion. The principal reason for the suitability of this approach is, as demonstrated in [13, Lemma 4.3] by making use of the generalised singular function, that the τ-measurable operators are continuous with respect to the restriction of this topology to the domain of the pertinent operator.

$q_{\mathcal{J}}$ is determined by norm boundedness, and the members of $\mathcal{J}^p$. In defining q_t we make adjustments to both of these determining factors. Firstly, it is clear that since the operators we consider are now unbounded, many of the sets of interest—for example, the images of norm-bounded sets—will in general be unbounded. Since the members of $\tilde{\mathcal{M}}$ are continuous with respect to the measure topology (when restricted to the domain of the operator) we can anticipate that boundedness in this topology will be more appropriate than norm boundedness. Following Nelson, we say that a set is bounded in measure if it is bounded in the topology of convergence in measure. Since $\alpha\mathcal{H}(\varepsilon, t) = \mathcal{H}(\alpha\varepsilon, t)$ for any $\alpha, \varepsilon, t > 0$, it follows that a set B is bounded in measure iff for all $t > 0$ there exists $\varepsilon > 0$ such that $B \subset \mathcal{H}(\varepsilon, t)$. Note that since $\mathcal{H}(\varepsilon, t_1) \subset \mathcal{H}(\varepsilon, t_2)$ for $t_1 \leq t_2$, it suffices to check this condition for small $t > 0$.

Secondly, we discriminate between projections by means of the trace. The following definitions are substantially the same as those appearing in [13, Sect. 4].

DEFINITION 3.1. (a) Suppose $t > 0$. Let $\mathcal{B}_t$ be the collection of all sets F satisfying the following conditions:

(i) F is bounded in measure

(ii) $\exists p \in \mathcal{M}^p$ such that $\tau(p) \leq t$ and $F \subset p\mathcal{H}$.

(b) For $B \subset \mathcal{H}$,

$$q_t(B) = \inf\{\varepsilon > 0 : \exists F \in \mathcal{B}_t \text{ such that } B \subset F + \varepsilon B_{\mathcal{H}}\}. \tag{19}$$

(c) Let $\mathcal{B}$ be the vector bornology $\bigcup_{t>0} \mathcal{B}_t$. The measure of non τ-precompactness of B is the $q_{\mathcal{B}}$-function relative to this bornology, i.e.,

$$q_\infty(B) = \inf\{\varepsilon > 0 : \exists F \in \mathcal{B} \text{ such that } B \subset F + \varepsilon B_{\mathcal{H}}\}. \tag{20}$$

(d) B is said to be τ-precompact if $q_\infty(B) = 0$.

Remark 3.2. It is clear that for any $B \subset \mathcal{H}$, the function $t \to q_t(B)$ is decreasing. It follows that $q_\infty(B) = \lim_{t\to\infty} q_t(B)$.

Note that since $\tau_{cm} \leq \|\cdot\|$, we have that norm boundedness implies boundedness in measure. Since the sum of two sets which are bounded in measure is again bounded in measure, it then follows from the definition that any set B for which $q_\infty(B) < \infty$ must be bounded in measure.

PROPOSITION 3.3. *If B is bounded in measure, then*

(a) $q_t(B) \leq \varepsilon$ *iff there exists* $p \in \mathcal{M}^p$ *such that* $\tau(p) \leq t$ *and* $B \subset p(B) + \varepsilon B_{\mathcal{H}}$.

(b) $q_\infty(B) \le \varepsilon$ *iff there exists* $p \in \mathcal{M}^p$ *such that* $\tau(p) < \infty$ *and* $B \subset p(B) + \varepsilon B_{\mathcal{H}}$.

Of course the proof of the above proposition follows the lines of Proposition 2.3—in this case we make an appeal to the measure continuity of p to show that the set $p(B)$ is bounded in measure. Therefore we omit the proof. It follows that for any $B \subset \mathcal{H}$

$$q_t(B) = \begin{cases} \inf\limits_{\substack{p\in\mathcal{M}^p \\ \tau(1-p)\le t}} \sup\limits_{\xi\in B} \|p\xi\| & \text{if } B \text{ is bounded in measure} \\ \infty & \text{otherwise} \end{cases} \tag{21}$$

$$q_\infty(B) = \begin{cases} \inf\limits_{\substack{p\in\mathcal{M}^p \\ \tau(1-p)<\infty}} \sup\limits_{\xi\in B} \|p\xi\| & \text{if } B \text{ is bounded in measure} \\ \infty & \text{otherwise.} \end{cases} \tag{22}$$

But these formulas are not valid for sets that are not bounded in measure: consider the set $p\mathcal{H}$, where $p \in \mathcal{M}^p$, $\tau(p) < \infty$.

Since any norm bounded set is bounded in measure, we have that $q_{\mathcal{M}_0}$ and q_∞ agree on sets that are bounded in norm, and that q_∞ takes on finite values on (some) sets that are bounded in measure but not norm bounded. As has been previously indicated this is very desirable as many sets arising in applications are bounded in measure but not in norm. Recall that for $B \subset \mathcal{H}$ we have that $q_{\mathcal{M}_0}(B) < \infty$ iff B is norm bounded. One might expect that $q_\infty(B) < \infty$ iff B is bounded in measure; unfortunately this is not the case, as shown by the following example. Thus the class of sets for which $q_\infty(B) < \infty$ is in general properly included in the class of those which are bounded in measure.

EXAMPLE 3.4. Let $\mathcal{H} = \mathcal{L}^2[0, \infty)$ and $\mathcal{M} = \mathcal{L}^\infty[0, \infty)$, where $[0, \infty)$ is equipped with the usual Borel σ-algebra and Lebesgue measure m.

Let $B = \{f_{n,\alpha} : 0 < \alpha \le 1, n \in \mathbf{N}\}$ where $f_{n,\alpha} = (1/\alpha)\chi_{[n,n+\alpha]}$; so $\|f_{n,\alpha}\|_2^2 = (1/\alpha)^2 \cdot \alpha = 1/\alpha$. If we are given $0 < t < 1$ then

$$(1 - \chi_{[n,n+t)})f_{n,\alpha} = \begin{cases} 0 & \text{if } t \ge \alpha \\ \dfrac{1}{\alpha}\chi_{[n+t,n+\alpha)} & \text{if } t < \alpha \end{cases}$$

and so $\|(1 - \chi_{[n,n+t)})f_{n,\alpha}\|_2^2 \le (1/\alpha)^2 \cdot |\alpha - t| \le 1/t^2$ for all n, α. Thus $B \subset \mathcal{H}(1/t, t)$, and so B is bounded in measure.

To show that $q_\infty(B) = \infty$, suppose we are given $A \in \Sigma$ such that $m(A) < \infty$. By (22) it suffices to show that for all $\varepsilon \geq 1$ there exists n, α such that $\|\chi_{A'} f_{n,\alpha}\|_2^2 \geq \varepsilon$.

Let $\alpha = 1/2\varepsilon$. Note that $m(A \cap [0, n)) \uparrow m(A)$ as $n \to \infty$. So we can choose $n \in \mathbf{N}$ such that $m(A \cap [n, \infty)) \leq \alpha/2$. In this case $m(A \cap [n, n + \alpha]) \leq \alpha/2$ and so $m(A' \cap [n, n + \alpha]) \geq \alpha/2$. Then $\|\chi_{A'} f_{n,\alpha}\|_2^2 = (1/\alpha)^2 \cdot m(A' \cap [n, n + \alpha]) \geq (1/\alpha)^2 \cdot \alpha/2 = 1/2\alpha = \varepsilon$ as required.

We now establish the analogues of Propositions 2.2 and 2.4 for the function q_t.

PROPOSITION 3.5. *Suppose B, B_1, $B_2 \subset \mathcal{H}$, $\lambda \in \mathbf{C}$.*

$$B_1 \subset B_2 \Rightarrow q_t(B_1) \leq q_t(B_2) \tag{23}$$

$$q_t(\lambda B) = |\lambda| q_t(B) \tag{24}$$

$$q_t(B_{\mathcal{H}}) = 1 \tag{25}$$

$$\max\{q_{t_1+t_2}(B_1); q_{t_1+t_2}(B_2)\} \leq q_{t_1+t_2}(B_1 \cup B_2) \tag{26}$$

$$q_{t_1+t_2}(B_1 \cup B_2) \leq \max\{q_{t_1}(B_1); q_{t_2}(B_2)\} \tag{27}$$

$$q_{t_1+t_2}(B_1 + B_2) \leq q_{t_1}(B_1) + q_{t_2}(B_2) \tag{28}$$

$$q_t(\overline{B}) = q_t(B). \tag{29}$$

Proof. These formulae are either obvious or follow in a similar way to those appearing in Proposition 2.4. For example, to show (27), suppose that for $i = 1, 2$ we have $p_i \in \mathcal{M}^p$ such that $\tau(p_i) \leq t_i$ and a measure bounded $F_i \subset p_i\mathcal{H}$ such that $B_i \subset F_i + \varepsilon_i B_{\mathcal{H}}$. Then the same argument as for (11) shows that

$$q_{t_1+t_2}(B_1 \cup B_2) \leq \max\{\varepsilon_1, \varepsilon_2\}$$

because $\tau(p_1 \vee p_2) \leq t_1 + t_2$.

The others follow similarly and so the proofs are omitted. ∎

By taking limits in the equations above we obtain

$$B_1 \subset B_2 \Rightarrow q_\infty(B_1) \leq q_\infty(B_2) \tag{30}$$

$$q_\infty(\lambda B) = |\lambda| q_\infty(B) \tag{31}$$

$$q_\infty(B_{\mathcal{H}}) = 1 \tag{32}$$

$$q_\infty(B_1 \cup B_2) = \max\{q_\infty(B_1); q_\infty(B_2)\} \tag{33}$$

$$q_\infty(B_1 + B_2) \leq q_\infty(B_1) + q_\infty(B_2) \tag{34}$$

$$q_\infty(\overline{B}) = q_\infty(B) \tag{35}$$

for $B, B_1, B_2 \subset \mathcal{H}$, $\lambda \in \mathbf{C}$. Some of these properties are immediate consequences of q_∞ being a $q_{\mathcal{B}}$-function.

As before we now consider the behaviour of q_t when applied to the images of sets under elements of $\tilde{\mathcal{M}}$. Equation (36) has been shown via a different method in [13, Theorem 4.4].

THEOREM 3.6. *Suppose $x \in \tilde{\mathcal{M}}$.*

(a) *Denote by $D_1(x)$ the set $D(x) \cap B_{\mathcal{H}}$. Then*

$$q_t(x[D_1(x)]) = \mu_t(x). \tag{36}$$

(b) *If $B \subset D(x)$ and $q_{t_2}(B) < \infty$ then, for any $t_1 > 0$,*

$$q_{t_1+t_2}(xB) \leq \mu_{t_1}(x) q_{t_2}(B). \tag{37}$$

(c) *Suppose that $y \in \tilde{\mathcal{M}}_0$, $B \subset D(x) \cap D(y)$, and $q_\infty(B) < \infty$. Then*

$$q_\infty(xB) = q_\infty((x + y)B). \tag{38}$$

Proof. (a) $x[D_1(x)]$ is bounded in measure, since this set is the continuous image of $D_1(x)$, which is bounded in measure. Thus by Eq. (21)

$$q_t(x[D_1(x)]) = \inf_{\substack{p \in \mathcal{M}^P \\ \tau(1-p) \leq t}} \sup_{\xi \in D_1(x)} \|px\xi\| = \inf_{\substack{p \in \mathcal{M}^P \\ \tau(1-p) \leq t}} \|px\| = \mu_t(x).$$

(b) We take $e = e_{\mu_{t_1}(x)}(|x^*|) \in \mathcal{M}^P$; then $\tau(1 - e) \leq t_1$ and $\mu_{t_1}(x) = \|ex\|$. Suppose we have $f \in \mathcal{M}^P$ such that $\tau(1 - f) \leq t_2$. We now imitate the proof of (16); the projection p constructed satisfies $\tau(1 - p) \leq t_1 + t_2$. As before

$$q_{t_1+t_2}(xB) \leq \sup_{\xi \in B} \|px\xi\| \leq \sup_{\xi \in B} \|p\| \, \|ex\| \, \|f\xi\| \leq \mu_{t_1}(x) \sup_{\xi \in B} \|f\xi\|$$

and the result follows.

(c) The proof of this is identical to that of (17). ∎

By taking limits in (36) and (37) we obtain

$$q_\infty(x[D_1(x)]) = \mu_\infty(x) \tag{39}$$

$$q_\infty(xB) \leq \mu_\infty(x) q_\infty(B) \tag{40}$$

for $x \in \tilde{\mathcal{M}}$, $B \subset D(x)$ such that $q_\infty(B) < \infty$. Note that this equation also shows that the latter class of sets is invariant under the operator, as is the subclass of τ-precompact sets that are included in the domain of the operator.

By combining Eqs. (39) and (40), we obtain the following corollary:

COROLLARY 3.7. *The following are equivalent:*

(a) $x \in \tilde{\mathcal{M}}_0$

(b) xB *is* τ*-precompact for every* $B \subset D(x)$ *such that* $q_\infty(B) < \infty$

(c) $x[D_1(x)]$ *is* τ*-precompact.*

As before (40) fails for arbitrary B included in the domain of the operator. Consider any $p \in \mathcal{M}^p$ such that $\tau(p) < \infty$. Then for any $t_1 > \tau(p)$ and any $t_2 > 0$ we have $q_{t_1+t_2}(p\mathcal{H}) = \infty$, but $\mu_{t_1}(p)q_{t_2}(\mathcal{H}) = 0 \cdot \infty = 0$. This is not a very interesting example, as the set $p\mathcal{H}$ is unbounded in every sense. The following example shows more: that (40) cannot be extended to the class of sets that are bounded in measure. In fact, we find a τ-compact operator that sends the set previously discussed in Example 3.4—bounded in measure but not having a finite q_∞ value—to a similar such set. Once again (40) fails by virtue of the agreement that $0 \cdot \infty = 0$.

EXAMPLE 3.8. Let $\mathcal{H}$, $\mathcal{M}$, and B be as in Example 3.4. Let

$$g = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \chi_{[n,n+1)}.$$

Then certainly $g \in \tilde{\mathcal{M}}$. To show that $q_\infty(g(B)) = \infty$ it suffices to show that if we are given $A \in \Sigma$ such that $m(A) < \infty$ then for all $\varepsilon \geq 1$ there exists $n, \alpha > 0$ such that $\|\chi_{A'} g f_{n,\alpha}\|_2^2 \geq \varepsilon$.

Choose $n \in \mathbf{N}$ such that $m(A \cap [n, n+1)) \leq 1/4\varepsilon n$. It is easy to see that this must be possible because of the divergence of the harmonic series. Put $\alpha = 1/2\varepsilon n$. Now $m(A \cap [n, n+\alpha]) \leq m(A \cap [n, n+1)) \leq 1/4\varepsilon n = \alpha/2$ and so $m(A' \cap [n, n+\alpha]) \geq \alpha/2$. Thus $\|\chi_{A'} g f_{n,\alpha}\|_2^2 = (1/\alpha)^2 \cdot (1/n) \cdot m(A' \cap [n, n+\alpha]) \geq (1/\alpha)^2 \cdot (1/n) \cdot (\alpha/2) = 1/2\alpha n = \varepsilon$ as required.

The above also provides the example promised after Theorem 2.5 by considering the bounded function g as a member of the norm closed ideal $\mathcal{M}_0$.

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